Econ 802

Answers to Final Exam

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1(a) Write a production plan as $y = (y_1 .. y_n)$ and prices as $p = (p_1 .. p_n) > 0$ where y_1 is output and p_1 is output price. Suppose in period 1 we observe (p^1, y^1) and in period 2 we observe (p^2, y^2). WAPM implies $p^1y^1 \geq p^1y^2$ and $p^2y^2 \geq p^2y^1$. Define $\Delta y \equiv y^2 - y^1$ so we have $p^2 \Delta y \geq 0$. Also, we have $-p^1y^2 + p^1y^1 \geq 0$ or $-p^1 \Delta y \geq 0$. Sum the inequalities and define $\Delta p = p^2 - p^1$. This gives $\Delta p \, \Delta y \geq 0$. If $\Delta p_1 > 0$ so output price rises, and $\Delta p_i = 0$ for $i = 2 .. n$, then it must be true that $\Delta p_1 \Delta y_1 \geq 0$, so $\Delta y_1 \geq 0$. This shows that output cannot fall.

(b) Define the profit function $\pi(p_1 .. p_n) \equiv \max \, py$ subject to $y \in Y$, where $y_1 \geq 0$ is output and $y_i \leq 0$ for $i = 2 .. n$ are inputs. From Hotelling's Lemma, $\frac{\partial \pi(p)}{\partial p_i} = y_i(p)$ for $i = 1 .. n$ where $y_i(p)$ is the net supply function. The Hessian of the profit function is

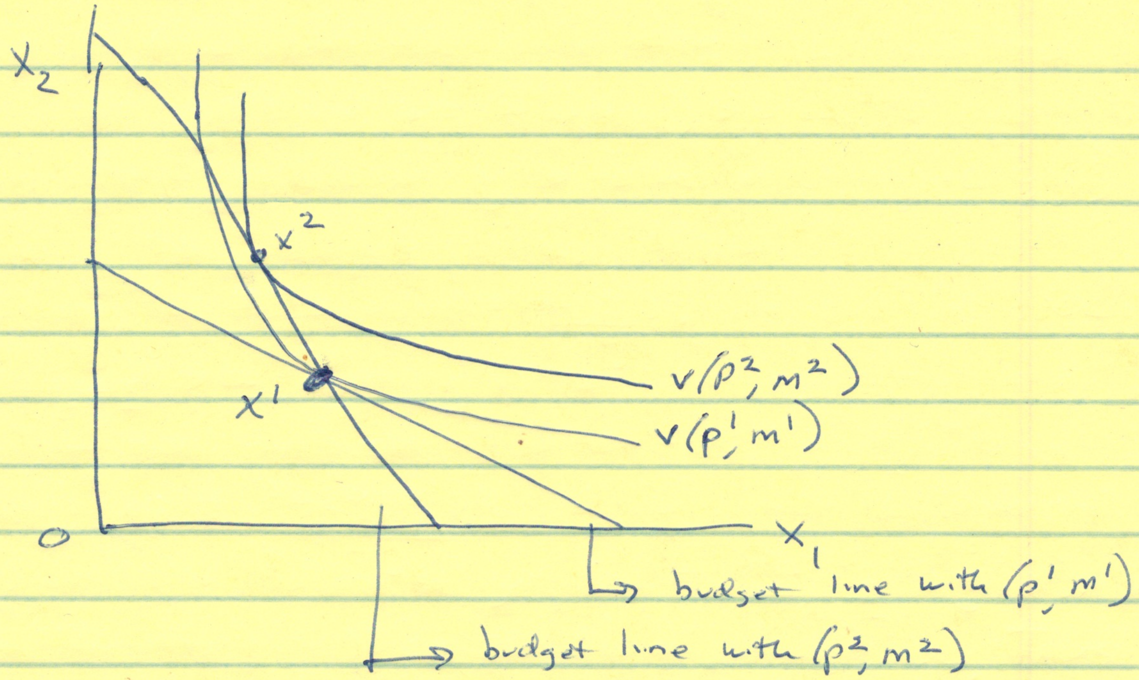
$$\frac{\partial^2 \pi(p)}{\partial p^2} = \begin{bmatrix} \frac{\partial y_1(p)}{\partial p_1} & \cdots & \frac{\partial y_1(p)}{\partial p_n} \\ \vdots & & \\ \frac{\partial y_n(p)}{\partial p_1} & \cdots & \frac{\partial y_n(p)}{\partial p_n} \end{bmatrix}$$

Because $\pi(p)$ is convex, its Hessian is positive semi-definite, which implies that the diagonal elements are non-negative. Thus $\frac{\partial y_1(p)}{\partial p_1} \geq 0$ so when p_1 rises, y_1 does not fall.

1(c) Let the firm's cost function be $c(w, y)$ where $y \geq 0$ is a scalar output and $w = (w_1 \ldots w_{n-1}) > 0$ are the input prices. The firm chooses y to solve $\max\{py - c(w, y)\}$ where $p > 0$ is output price. Assume an interior solution. The FOC is $p = \frac{\partial c(w, y)}{\partial y}$. The sufficient SOC is $\frac{\partial^2 c(w, y)}{\partial y^2} > 0$. Let $y(p)$ be the firm's output supply function and substitute it into FOC to get the identity $p \equiv \frac{\partial c[w, y(p)]}{\partial y}$. Differentiate with respect to p on both sides to get $1 = \frac{\partial^2 c[w, y(p)]}{\partial y^2} y'(p)$. From the sufficient SOC this implies $y'(p) > 0$. Thus when p increases, so does y.

2(a) In general, we expect that when a price rises, the consumer is worse off, or at least no better off, because the feasible set cannot expand. We also expect that when income rises, the consumer is better off, or at least no worse off, because the feasible set becomes larger. The indirect utility function depends only on the ratios m/p_i because it must be homogeneous of degree zero in (p, m). This occurs because multiplying all prices and income by the same positive constant $t > 0$ does not change the feasible set. Therefore it does not change the optimal bundle or the maximum utility level. So all of the features of $V(p, m)$ make economic sense.

(b) I would expect that in most cases $V(p^2, m^2) > V(p^1, m^1)$, although this depends on some assumptions.
Here is the graphical argument:



In period 1, Asha chooses x^1, ~~which~~ where the budget line is tangent to an indifference curve. This gives utility $v(p^1, m^1)$. In period 2, the prices p^2 give a budget line with a new slope. Suppose it is steeper than p^1. If m^2 is large enough that she can still afford x^1, there will be feasible points in period 2 that are on indifference curves above x^1. Thus she chooses some point like x^2 and gets $v(p^2, m^2) > v(p^1, m^1)$. Normally we get a result of this kind whenever we have interior solutions and strictly convex preferences.

A similar argument applies if the new budget line is flatter.

2(c) We use $u(x) = \min_p v(p, 1)$ subject to $px = 1$

The Lagrangean is

$$L = \sum_i a_i \ln\left(\frac{a_i}{p_i}\right) + \lambda\left[\sum_i p_i x_i - 1\right]$$

Note: we are setting $m = 1$ here.

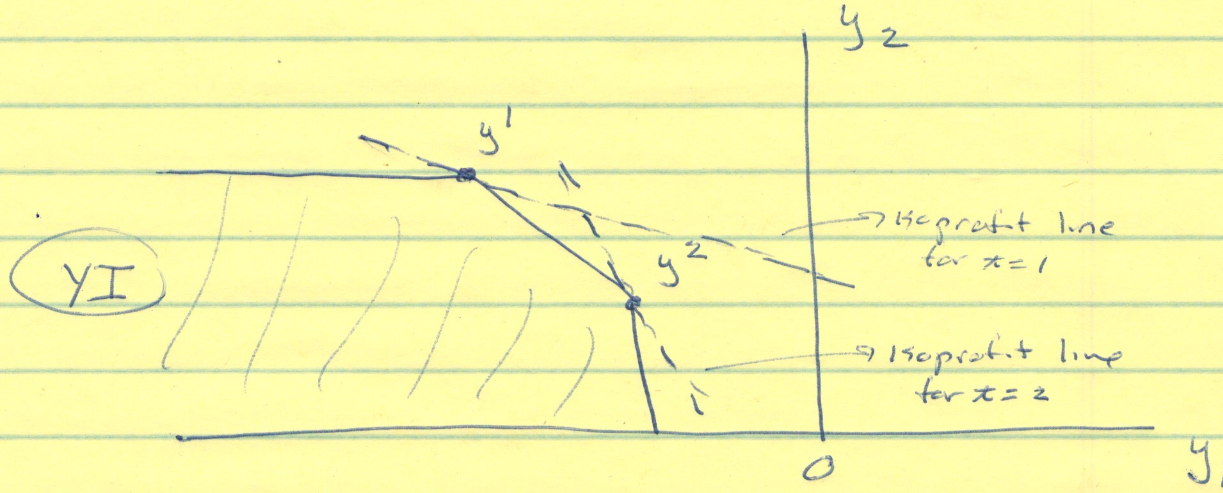
FOC: $\dfrac{\partial L}{\partial p_i} = a_i\left(\dfrac{p_i}{a_i}\right)\left(-\dfrac{a_i}{p_i^2}\right) + \lambda x_i = 0 \quad i = 1 \ldots n$

or $\lambda x_i = \frac{a_i}{p_i}$, $i = 1..n$. This gives $\lambda p_i x_i = a_i$, all i.
Sum these equations to get $\lambda \sum_i p_i x_i = \sum_i a_i$
where $\sum_i p_i x_i = 1$ from the constraint and
$\sum_i a_i = 1$ by assumption.
Thus $\lambda = 1$ and $p_i = \frac{a_i}{x_i}$ for $i = 1..n$.
Substitute these results into $v(p,1)$ to get

$$u(x) = \sum_i a_i \ln\left(\frac{a_i}{p_i}\right) = \sum_i a_i \ln x_i$$

[Note: This is a log transformation of the Cobb-Douglas utility function $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$]

3(a)



Because the firm is profit-maximizing, y^2 must be on or below the isoprofit line through y^1 in period 1, and y^1 must be on or below the isoprofit line through y^2 in period 2. Because YI is convex, it must include all points on the line segment between y^1 and y^2. Because it is monotonic, it must include all points along the horizontal line to the left of y^1, all points along the vertical line below y^2, and all points in the shaded area. Because it is closed, it includes all boundary points.

3(a) continued

Yes, if Y^I is the true PPS, then there is always a solution to the profit max problem. From $(p_1, p_2) > 0$, the isoprofit lines always have a finite negative slope. The highest isoprofit line will be reached at y^1, y^2, or all points on the line segment between them, depending on the price ratio $\frac{p_1}{p_2}$.

(b) There are two ways to show this. (either one is fine, you don't need both)
The fast method is to notice that CRS implies a cost function of the form $c(w,y) = c(w,1)y$.
Shephard's Lemma gives
$$x_1(w,y) = \frac{\partial c(w,1)y}{\partial w_1}$$
$$x_2(w,y) = \frac{\partial c(w,1)y}{\partial w_2}$$

Divide the second equation by the first to get
$$\frac{x_2(w,y)}{x_1(w,y)} = \frac{\frac{\partial c(w,1)}{\partial w_2}}{\frac{\partial c(w,1)}{\partial w_1}} \Bigg\} \text{ independent of output } y$$

This shows that the ratio of conditional input demands is constant for all $y > 0$, so we get a graph like this:

where the curves are isoquants and the lines are isocost lines

Here is the slower method:

Suppose $x^* = (x_1^*, x_2^*)$ solves the cost minimization problem for $y^* > 0$. This implies $y^* = f(x^*)$ and from FOC we have

$$\frac{\frac{\partial f(x^*)}{\partial x_1}}{\frac{\partial f(x^*)}{\partial x_2}} = \frac{w_1}{w_2}$$

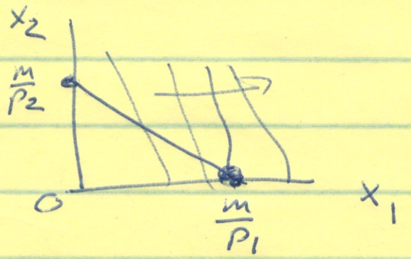
Now suppose the firm wants the output ty^* where $t > 0$. Due to CRS, this can be obtained from (tx_1^*, tx_2^*) because $f(tx^*) = tf(x^*) = ty^*$.

Also due to CRS, $f(x)$ is homogeneous of degree $k=1$, so its derivatives are homogeneous of degree $k-1=0$. Thus

$$\frac{\frac{\partial f(tx^*)}{\partial x_1}}{\frac{\partial f(tx^*)}{\partial x_2}} = \frac{w_1}{w_2}$$

So tx^* satisfies the FOC for cost min. Because $\frac{tx_2^*}{tx_1^*} = \frac{x_2^*}{x_1^*}$, the ratio of inputs remains the same, and we get the same graph as before.

3(c) (1) If the indifference curves are steeper than the budget line, we get a graph like this:



So $x_1(p, m) = \frac{m}{p_1}$

and $x_2(p, m) = 0$

$\Rightarrow v(p, m) = \frac{am}{p_1}$

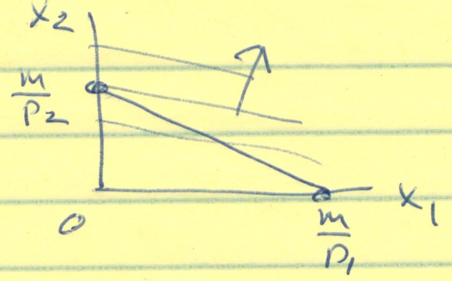
(2) If the indifference curves are flatter than the budget line, we get a graph like this:

3(c) continued

so $x_1(p,m) = 0$

$x_2(p,m) = \frac{m}{p_2}$

$\Rightarrow V(p,m) = \frac{bm}{p_2}$



③ In the case where the indiff. curves and budget line have the same slope, all points on the budget line (including the boundary points) are optimal, so

$$V(p,m) = \frac{am}{p_1} = \frac{bm}{p_2}$$

If you use Roy's identity in case ①, you will get

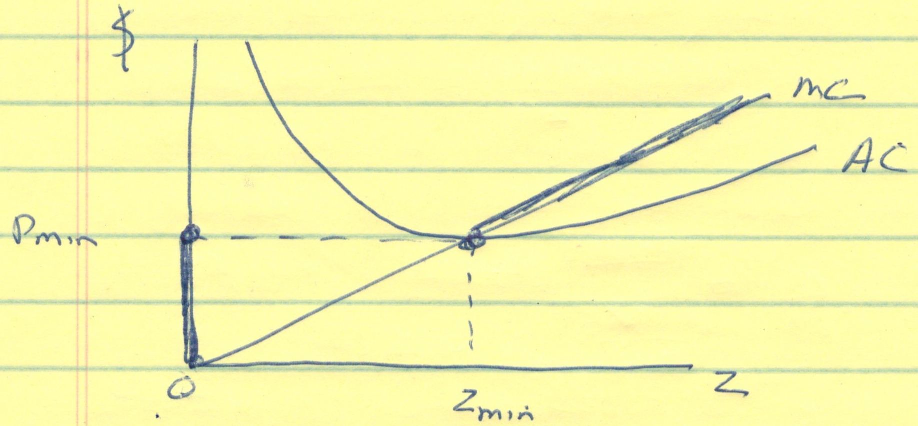
$$\left.\begin{aligned} x_1(p,m) &= -\frac{\frac{\partial V(p,m)}{\partial p_1}}{\frac{\partial V(p,m)}{\partial m}} = \frac{m}{p_1} \\ x_2(p,m) &= 0 \end{aligned}\right\} \text{correct.}$$

Likewise if you use Roy's identity in case ② you will get

$$\left.\begin{aligned} x_1(p,m) &= 0 \\ x_2(p,m) &= -\frac{\frac{\partial v(p,m)}{\partial p_2}}{\frac{\partial v(p,m)}{\partial m}} = \frac{m}{p_2} \end{aligned}\right\} \text{correct.}$$

But even though the indirect utility function is continuous, in case ③ it is not possible to use Roy's identity. The problem is that there are multiple solutions to the utility max problem and the Marshallian demands are not well-defined. Furthermore, arbitrarily small changes in a price can lead to discontinuous jumps in consumer behavior.

4(a) Consider the cost curves for an individual firm. We have $MC = 2bz$ and $AC = \frac{a}{z} + bz$. The AC curve reaches a minimum where $-\frac{a}{z^2} + b = 0$. Call the solution of this FOC $z_{min} = \left(\frac{a}{b}\right)^{1/2}$. To find the level of AC when it hits the minimum, substitute: $AC(z_{min}) = a\left(\frac{b}{a}\right)^{1/2} + b\left(\frac{a}{b}\right)^{1/2} = 2(ab)^{1/2}$. Call this $p_{min} = 2(ab)^{1/2}$. The cost curves are like this:



The firm's supply function is the heavy shaded curve with a jump at p_{min}:

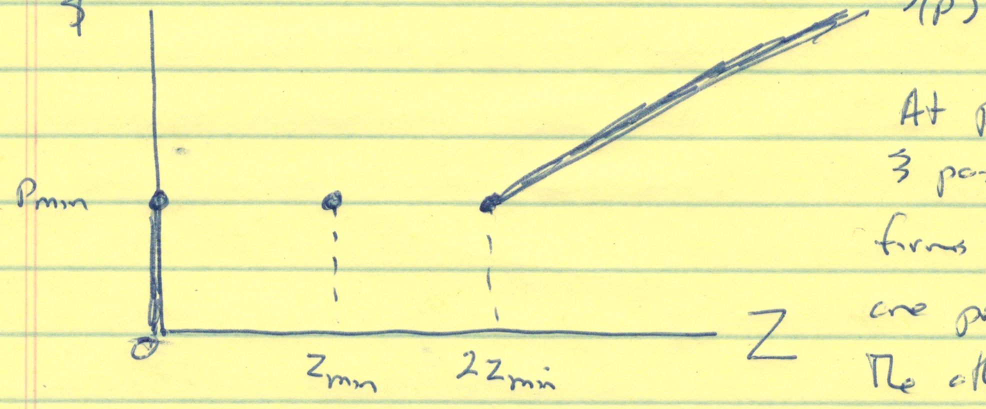
For $p < p_{min}$, $z(p) = 0$

For $p = p_{min}$, $z(p) = 0$ or z_{min} (both are optimal and give zero profit)

For $p > p_{min}$, we have

$p = MC = 2bz$, so $z(p) = \frac{p}{2b}$ (with positive profit)

With two firms, the market supply looks like this:



At p_{min} there are 3 possibilities: both firms produce zero, one produces z_{min} and the other produces zero, or both produce z_{min}.

4(a) continued

The consumer maxes $hx^{1/2} + y$ subject to $px + y = w$

or $hx^{1/2} + w - px$. FOC: $\frac{h}{2}x^{-1/2} - p = 0$

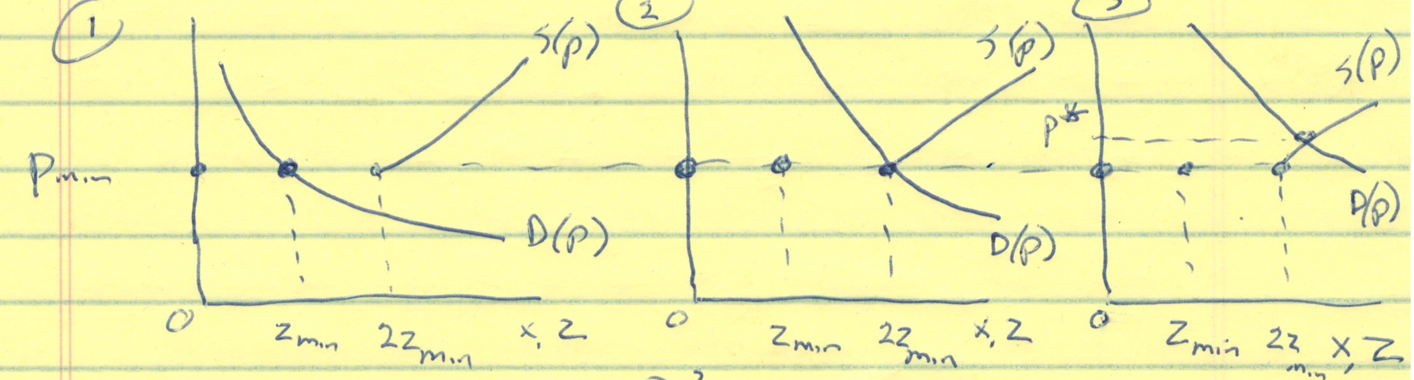
(SOC sufficient holds) $\Rightarrow \frac{h}{2} = px^{1/2} \Rightarrow x^{1/2} = \frac{h}{2p}$

So the market demand function is

$$D(p) = \left(\frac{h}{2p}\right)^2.$$

(b) The demand function does not have a finite vertical intercept. ① If $D(p_{min}) = z_{min}$, we can have an equilibrium where one firm produces z_{min} and the other produces nothing (each firm is maximizing profit).

② If $D(p_{min}) = 2z_{min}$, we can have an equilibrium where both firms produce z_{min}. ③ If $D(p_{min}) > 2z_{min}$ we have an intersection where $D(p^*) = S(p^*)$ for some $p^* > p_{min}$. The graphs look like this:



Case ① requires $\left(\frac{h}{2p_{min}}\right)^2 = z_{min}$

$$\Rightarrow \frac{h}{2\cdot 2(ab)^{1/2}} = \left(\frac{a}{b}\right)^{1/4} \Rightarrow h = 4(ab)^{1/2}\left(\frac{a}{b}\right)^{1/4}$$

Case ② requires

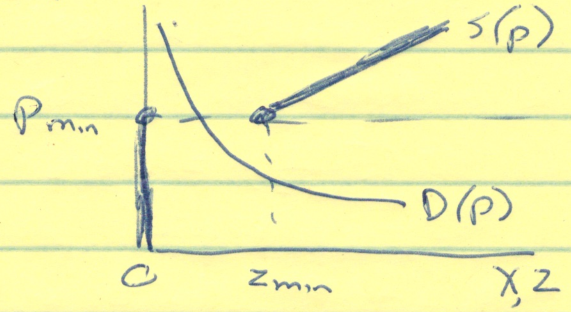
$$\left(\frac{h}{2p_{min}}\right)^2 = 2z_{min}$$

$$\Rightarrow \frac{h}{2\cdot 2(ab)^{1/2}} = 2^{1/2}\left(\frac{a}{b}\right)^{1/4} \Rightarrow h = 2^{5/2}(ab)^{1/2}\left(\frac{a}{b}\right)^{1/4}$$

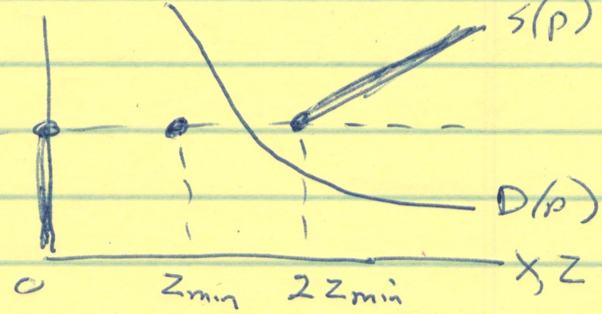
Case ③ occurs iff h is larger than in case ②.

4(b) continued

At any other value of h, $D(p)$ passes through a gap in the market supply function and we get either



or



In cases like this, $P < P_{min} \Rightarrow$ excess demand
$P > P_{min} \Rightarrow$ excess supply

$P = P_{min} \Rightarrow$ at least one firm must produce a positive output less than z_{min}. This implies negative profit, which contradicts profit max (zero profit is feasible).

(c). If we thought about this as a GE model, we would need to include the profits of the firms (if any) on the right hand side of the consumer's budget constraint. This is not a big issue. The real problem is that the aggregate excess demand function would need to have three properties to guarantee existence of a WE: continuity, zero homogeneity, and Walras's Law. The last two would be satisfied, but the gap in the market supply function for the x good creates a discontinuity. Thus we cannot use the method of proof in Varian. But keep in mind that the 3 conditions mentioned above are <u>sufficient</u> for existence, not <u>necessary</u>. Even with a discontinuity, we can sometimes get a WE if we happen to have the right parameter values.

5(a) A has Leontief preferences and maximizes utility by setting $\alpha x_{A1} = \beta x_{A2}$. The budget constraint is $p_1 x_{A1} + p_2 x_{A2} = m_A$.

Substitute $x_{A2} = \left(\frac{\alpha}{\beta}\right) x_{A1} \Rightarrow p_1 x_{A1} + p_2 \left(\frac{\alpha}{\beta}\right) x_{A1} = m_A$

$$\Rightarrow \boxed{x_{A1}(p, m_A) = \frac{m_A}{p_1 + p_2\left(\frac{\alpha}{\beta}\right)}}$$ Marshallian demand for good 1

B has the Lagrangean $L = x_{B1}^{1/2} x_{B2}^{1/2} - \lambda\left[p_1 x_{B1} + p_2 x_{B2} - m_B\right]$

FOC:

$$\left.\begin{aligned} \tfrac{1}{2} x_{B1}^{-1/2} x_{B2}^{1/2} &= \lambda p_1 \\ \tfrac{1}{2} x_{B1}^{1/2} x_{B2}^{-1/2} &= \lambda p_2 \end{aligned}\right\} \Rightarrow \frac{x_{B2}}{x_{B1}} = \frac{p_1}{p_2}$$

or $x_{B2} = \left(\frac{p_1}{p_2}\right) x_{B1}$. Substitute into budget constraint:

$$p_1 x_{B1} + p_2\left(\frac{p_1}{p_2}\right) x_{B1} = m_B \Rightarrow \boxed{x_{B1}(p, m_B) = \frac{m_B}{2p_1}}$$

Marshallian demand for good 1

From the endowments,

$$m_A = p_2 w_2 \quad \text{and} \quad m_B = p_1 w_1$$

So market demand for good 1 is

$$x_{A1} + x_{B1} = \frac{p_2 w_2}{p_1 + p_2\left(\frac{\alpha}{\beta}\right)} + \frac{p_1 w_1}{2p_1}$$

$$= \frac{w_2}{\left(\frac{p_1}{p_2}\right) + \left(\frac{\alpha}{\beta}\right)} + \frac{w_1}{2}$$

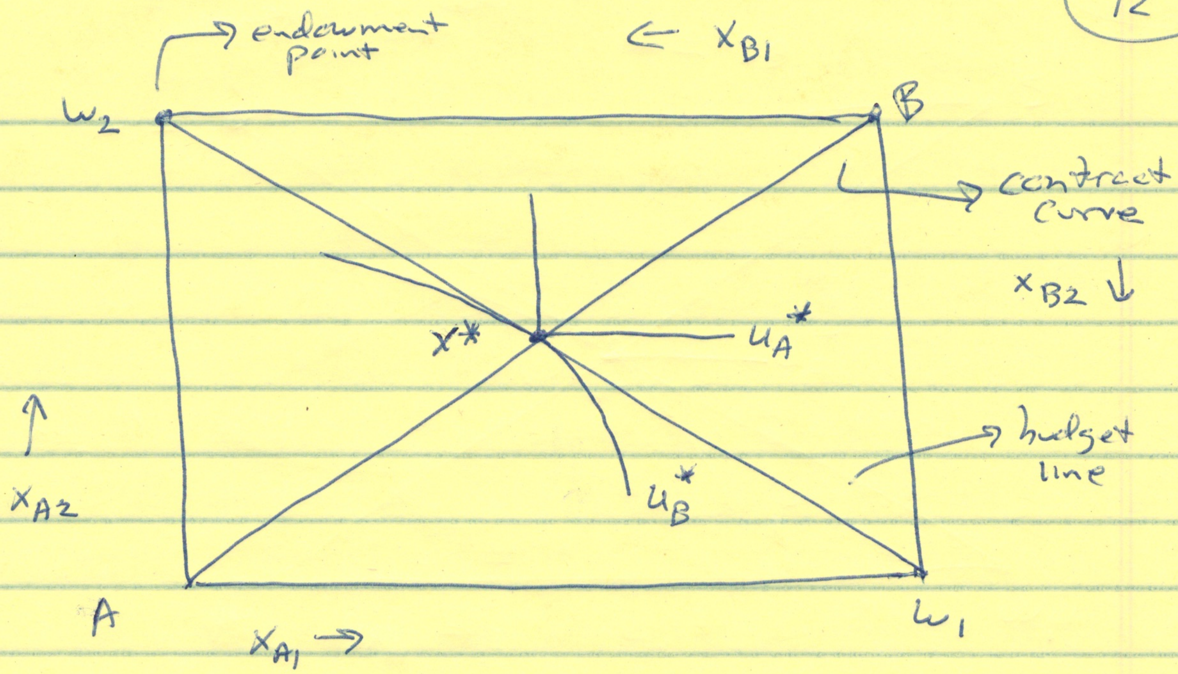
Set this equal to the supply of good 1 (which is w_1) and solve for $\frac{p_1}{p_2}$: $\frac{w_2}{\left(\frac{p_1}{p_2}\right) + \left(\frac{\alpha}{\beta}\right)} = \frac{w_1}{2}$

$\Rightarrow \frac{2w_2}{w_1} = \frac{p_1}{p_2} + \frac{\alpha}{\beta}$. Now use the assumption $\alpha w_1 = \beta w_2 \Rightarrow \frac{\alpha}{\beta} = \frac{w_2}{w_1}$, which gives

$$\boxed{\frac{p_1}{p_2} = \frac{w_2}{w_1}}$$

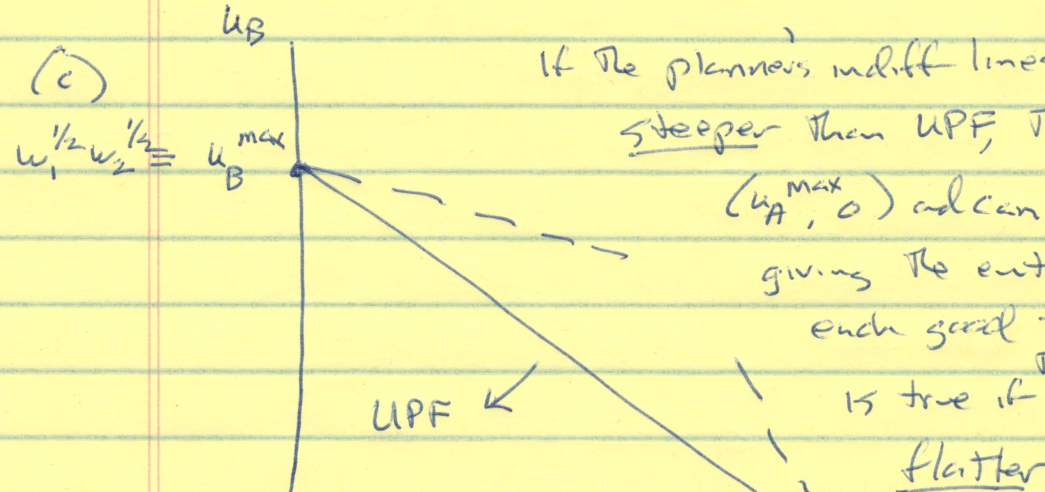
(as usual, we can only solve for a ratio and we can ignore the market for good 2 due to Walras's Law)

5(b)



The endowment point is the upper left corner, the budget line goes from the endowment to the lower right corner (this follows from $\frac{p_1}{p_2} = \frac{w_2}{w_1}$), the contract curve is the line from A's origin to B's origin (this follows from $\alpha w_1 = \beta w_2$), and the equilibrium allocation is x^*. This is Pareto efficient because any other point that gives $u_A \geq u_A^*$ must give $u_B < u_B^*$, and any other point that gives $u_B \geq u_B^*$ must give $u_A < u_A^*$.

(c)



If the planner's indiff lines (dashed) are <u>steeper</u> than UPF, the planner wants $(u_A^{max}, 0)$ and can achieve this by giving the entire endowment of each good to A. The opposite is true if the planner's indiff lines are <u>flatter</u> than UPF. If the slopes are equal, the planner is indifferent.